A NOTE ON THE CALCULATION OF EULER'S CONSTANT.

By GOLDIE HORTON, University of Texas.

Euler's constant, which plays an important rôle in the theory of gamma functions, is usually defined by the relation

$$\gamma = \lim_{n=\infty} \sum_1^n \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right) \right].$$

Its direct calculation from this definition is impracticable, and it is actually computed by means of the asymptotic expansion

$$\gamma = \lim_{n=\infty} \sum_1^n \frac{1}{m} - \log n - \frac{1}{2n} + \frac{B_1}{2n^2} - \frac{B_2}{2n^4} + \cdots, \tag{1}$$

where the B's are the Bernoullian numbers, the error being less than the first term omitted.[1] The proof of (1) is a delicate matter, however, and it seems therefore worth while to call attention to the fact that γ can be calculated as readily as π or $\log n$ to three or four places of decimals, if one makes use of an idea employed in Cauchy's integral test. This method, which is applicable to all convergent series of positive monotone decreasing terms, depends upon the fact that from Cauchy's test it follows immediately that both

$$\sum_1^n u_m + \int_n^\infty u_m dm \qquad \text{and} \qquad \sum_1^n u_m + \int_{n+1}^\infty u_m dm$$

approximate $\sum_1^\infty u_m$ with an error less than $\int_n^{n+1} u_m dm$, and hence less than u_n.[1]

Applying this to

$$\gamma = \sum_1^\infty \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right) \right],$$

we see that

$$\sum_1^n \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right) \right] + \int_n^\infty \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right) \right] dm$$

or

$$\sum_1^n \left[\frac{1}{m} - \log\left(1 + \frac{1}{m}\right) \right] + \left[-1 - \log\left(\frac{n}{n+1}\right)^{n+1} \right]$$

approximates to Euler's constant with an error less than

$$\log\left(\frac{n+1}{n+2}\right)^{n+2} \left(\frac{n+1}{n}\right)^{n+1}$$

For $n = 5$, this gives $\gamma = .58249$ with an error less than .01487; for $n = 10$, $\gamma = .57948$ with an error less than .00429; for $n = 20$, $\gamma = .57779$ with an error less than .00115.

[1] See Wm. Shank's paper, "On the Calculation of the Numerical Value of Euler's Constant." *Proceedings of the Royal Society of London*, Vol. XV, p. 429.

[1] In these integrals the definition of the function u_m is extended so that it relates to the continuous variable m in such wise that u_m is a monotone decreasing function of the continuous variable m.